ON SOME STATISTICS COMPARING TWO BINOMIAL SEQUENCES

BY P. V. KRISHNA IYER AND M. N. BHATTACHARYYA

Defence Science Laboratory, New Delhi

I. INTRODUCTION

FOR two sequences of observations $\{x\}$ and $\{y\}$ given by

$$\{x\}—x_1, x_2, x_3, \ldots, x_n$$
$$\{y\}—y_1, y_2, y_3, \ldots, y_n$$

where x's and y's take any of the values $\theta_1, \theta_2, \ldots, \theta_k$ with probabilities $p_1, p_2, \ldots, p_k$, Iyer (1954) has discussed the probability distributions of the *Statistics* $\Sigma(x_r - y_r)$, $\Sigma|x_r - y_r|$ and $\Sigma(x_r - y_r)^2$ arising from considerations of simple matchings and suggested their application for testing the homogeneity of the two sequences. In an earlier note published in *Nature* (1953) it was suggested that the probability distribution of the *Statistics*

$$\left.\begin{array}{l}\Sigma|x_r-y_r| + \Sigma|x_{r+1}-y_r| + \ldots + \Sigma|x_{r+s}-y_r| \\ + \Sigma|x_r-y_{r+1}| + \Sigma|x_r-y_{r+2}| + \ldots \Sigma|x_r-y_{r+s}|\end{array}\right\}$$

can be studied to find out an optimum value of s for the purpose. Following the above lines we formulate three *Statistics* X_s, Y_s and Z_s defined as

$$X_s = \sum_{r=1}^{n}(x_r - y_r) + \sum_{i=1}^{s}\sum_{r=1}^{n-i}(x_r - y_{r+i}) + \sum_{i=1}^{s}\sum_{r=1}^{n-i}(x_{r+i} - y_r),$$

$$Y_s = \sum_{r=1}^{n}|x_r - y_r| + \sum_{i=1}^{s}\sum_{r=1}^{n-i}|x_r - y_{r+i}| + \sum_{i=1}^{s}\sum_{r=1}^{n-i}|x_{r+i} - y_r|$$

and

$$Z_s = \sum_{r=1}^{n}(x_r - y_r)^2 + \sum_{i=1}^{s}\sum_{r=1}^{n-i}(x_r - y_{r+i})^2 + \sum_{i=1}^{s}\sum_{r=1}^{n-i}(x_{r+i} - y_r)^2,$$

where s can take values from 0 to $(n - 1)$.

The probability distributions of these *Statistics* can be studied with a view to finding an efficient test for the purpose of comparing two sequences. It may be noted that X_0, Y_0 and Z_0 correspond to $\Sigma(x_r - y_r)$, $\Sigma|x_r - y_r|$ and $\Sigma(x_r - y_r)^2$ respectively. The first two moments of these three *Statistics* have been calculated by Iyer (1954) and he has stated that all the cumulants are linear functions in n, the size of the sample and hence as n increases the distributions will take the normal form. The distributions of the *Statistics* X_s, Y_s and Z_s for $s > 0$ are under investigation.

The discussions of the above *Statistics* for two binomial sequences are comparatively simple, but do not appear to have been dealt with so far. In connection with a psychological investigation McNemar and Cochran have discussed a method for comparing the percentages in matched samples. For two matched samples involving binomial characters A and B, if b and c are the number of matched combinations (AB) and (BA) then the test proposed by them consists in calculating

$$\chi^2 = \frac{(b-c)^2}{(b+c)} \text{ (d.f. = 1).}$$

This involves the assumption that the probability of matched combination (AB) or (BA) is $\frac{1}{2}$. But this test is different from the *Statistics* X_0, Y_0 and Z_0 which are also based on simple matchings.

The object of the present paper is to discuss the probability distributions of the *Statistics* X_s, Y_s and Z_s for the case of two binomial sequences, where x's and y's are either A or B with probabilities p and q respectively, by adopting the following system of scoring.

$$(x-y) = \left.\begin{array}{l} (A-B) \\ = (B-A) \\ = \begin{pmatrix} (A-A) \\ \text{or} \\ (B-B) \end{pmatrix} \end{array}\right\} \begin{array}{c} +1 \\ -1 \\ 0 \end{array} \quad |x-y| = \left.\begin{array}{l} |A-B| \\ = |B-A| \\ = \begin{pmatrix} |A-A| \\ \text{or} \\ |B-B| \end{pmatrix} \end{array}\right\} \begin{array}{c} +1 \\ +1 \\ 0 \end{array} \quad (x-y)^2 = \left.\begin{array}{l} (A-B)^2 \\ = (B-A)^2 \\ = \begin{pmatrix} (A-A)^2 \\ \text{or} \\ (B-B)^2 \end{pmatrix} \end{array}\right\} \begin{array}{c} +1 \\ +1 \\ 0 \end{array}$$

Obviously, according to the scoring system adopted, the *Statistics* Y_s and Z_s are the same. Hence we shall confine ourselves in discussing either of the two only. The purpose of discussing such *Statistics* is to develop a method of comparing two binomial samples on the basis of the order of occurrence of the individual observations or in other words comparing two binomial sequences.

As is seen from above, these *Statistics* are based on a footing entirely different from that of the usual 2×2 contingency x^2 used for comparing two binomial samples. X_s and Y_s do not exist when the order of observations is not maintained. However under some assumptions they are comparable with the X^2 test. Even in that case because of the inclusions of additional information regarding the order, these *Statistics* are reasonably expected to provide more reliable tests than the x^2, where the order is completely ignored. The actual applications of these *Statistics* and their powers *vis-a-vis* 2×2 contingency x^2 will be the subject of a separate paper.

2. Probability Generating Functions and Cumulants of the Distributions

A. (*a*) *Statistic* X_0

$$X_0 = \sum_{r=1}^{n} (x_r - y_r) \tag{A. a. 1}$$

According to the system of scoring defined earlier, X_0 is the number of positive differences $(A - B)$ *minus* the number of negative differences $(B - A)$ observed by matching the individuals of the samples on the basis of the order of observations.

The probability generating function (P.G.F.) $\phi(n)$ for the Statistic X_0 is

$$\phi(n, X_0) = [p^2 + q^2 + pqt + pqt^{-1}]^n \tag{A.a.2}$$

The first four cumulants of this distribution can be readily calculated and they are as follows:—

$$\kappa_1 = 0,\ \kappa_2 = n\,2pq,\ \kappa_3 = 0,\ \kappa_4 = n\,2pq\,(1 - 6pq) \tag{A.a.3}$$

(*b*) *Statistic* X_1

$$X_1 = \sum_{r=1}^{n} (x_r - y_r) + \sum_{r=1}^{n-1} (x_r - y_{r+1}) + \sum_{r=1}^{n-1} (x_{r+1} - y_r) \tag{A.b.1}$$

The distribution of X_1 is more complicated than that of X_0. We shall discuss this distribution by obtaining the recurrence relation for the P.G.F.S.

Assuming $P_{\binom{A}{A}}(n, r)$, $P_{\binom{A}{B}}(n, r)$, $P_{\binom{B}{A}}(n, r)$ and $P_{\binom{B}{B}}(n, r)$ to be the probabilities of X_1 taking a value r for n observations according as the first observation of the two sequences are as given in the suffix, the following relations can be easily established.

$$P_{\binom{A}{A}}(n, r) = p^2 P_{\binom{A}{A}}(n-1, r) + p^2 P_{\binom{A}{B}}(n-1, r-1) + p^2 P_{\binom{B}{A}}(n-1, r+1) + p^2 P_{\binom{B}{B}}(n-1, r)$$

$$P_{\binom{A}{B}}(n, r) = pqP_{\binom{A}{A}}(n-1, r-2) + pqP_{\binom{A}{B}}(n-1, r-3) + pqP_{\binom{B}{A}}(n-1, r-1) + pqP_{\binom{B}{B}}(n-1, r-2)$$

$$P_{\binom{B}{A}}(n, r) = pqP_{\binom{A}{A}}(n-1, r+2) + pqP_{\binom{A}{B}}(n-1, r+1) + pqP_{\binom{B}{A}}(n-1, r+3) + pqP_{\binom{B}{B}}(n-1, r+2)$$

$$P_{\binom{B}{B}}(n, r) = q^2 P_{\binom{A}{A}}(n-1, r) + q^2 P_{\binom{A}{B}}(n-1, r-1) + q^2 P_{\binom{B}{A}}(n-1, r+1) + q^2 P_{\binom{B}{B}}(n-1, r)$$

(A.b.2)

Reducing these equations in terms of P.G.F.'s and eliminating we get

$$\begin{vmatrix} (E-p^2)\,t & -p^2t^2 & -p^2 & -p^2t \\ -pqt^2 & (E-pqt^3) & -pqt & -pqt^2 \\ -pqt & -pqt^2 & (Et^3-pq) & -pqt \\ -q^2t & -q^2t^2 & -q^2 & (E-q^2)\,t \end{vmatrix} \phi(n, X_i) = 0$$

(A.b.3)

where $\phi(n, X_1)$ is the P.G.F. of X_1 for n observations in each of the sequences.

Solution of the above determinant reduces (A.b.3) to

$$[E^4 - E^3\{p^2 + q^2 + pqt^3 + pqt^{-3}\}]\,\phi(n, X_1) = 0 \qquad \text{(A.}b\text{.4)}$$

or

$$\phi(n+4, X_1) - \phi(n+3, X_1)\{p^2+q^2+pqt^3+pqt^{-3}\} = 0 \qquad \text{(A.}b\text{.5)}$$

For $n = 1$

$$\phi(5, X_1) - \phi(4, X_1)\{p^2+q^2+pqt^3+pqt^{-3}\} = 0 \qquad \text{(A.}b\text{.6)}$$

On substitution of the directly enumerated values of $\phi(5, X_1)$ and $\phi(4, X_1)$ the left-hand side of (A.b.6) has been found to vanish.

By putting $t = e^{\theta}$ in (A.b.4) the difference equation connecting the moment generating functions (M.G.F.'s) is obtained.

$$[E^4 - E^3\{p^2 + q^2 + pqe^{3\theta} + pqe^{-3\theta}\}]\, M(n, X_1) = 0 \qquad \text{(A.}b\text{.7)}$$

We note that the characteristic equation of ($A.b$.7) has only one non-vanishing root $\lambda = \{p^2 + q^2 + pqe^{3\theta} + pqe^{-3\theta}\}$ which is unity when $\theta = 0$. Then

$$M(n, X_1) = A_1\lambda^n \qquad \text{(A.}b\text{.8)}$$

where A_1 is a constant independent of n and can be written as $M(2, X_1)/\lambda^2$, where

$$M(2, X_1) = (p^4 + 4p^2q^2 + q^4) + 2pq\,(p^2 + q^2)\,(e^{2\theta} + e^{-2\theta}) + p^2q^2\,(e^{4\theta} + e^{-4\theta}) \qquad \text{(A.}b\text{.9)}$$

Thus

$$M(n, X_1) = M(2, X_1)\,\lambda^{n-2} \qquad \text{(A.}b\text{.10)}$$

The rth cumulant of the distribution is

$$\kappa_r = \left[\frac{\partial^r}{\partial\theta^r}\{\log M(2, X_1) + (n-2)\log\lambda\}\right]_{\theta=0} \qquad \text{(A.}b\text{.11)}$$

Differentiating $M(2, X_1)$ and λ in succession and putting $\theta = 0$, we get

$$\left.\begin{array}{ll} M^{\text{I}}(2, X_1) = 0 & \lambda^{\text{I}} = 0 \\ M^{\text{II}}(2, X_1) = 16pq & \lambda^{\text{II}} = 18pq \\ M^{\text{III}}(2, X_1) = 0 & \lambda^{\text{III}} = 0 \\ M^{\text{IV}}(2, X_1) = 64pq\,(1 + 6pq) & \lambda^{\text{IV}} = 162pq \end{array}\right\} \qquad \text{(A.}b\text{.12)}$$

using (A.b.11) we obtain

$$\left.\begin{array}{ll} \kappa_1 = 0 & \kappa_3 = 0 \\ \kappa_2 = (9n-10)\,2pq & \kappa_4 = (81n-130)\,2pq\,(1-6pq) \end{array}\right\} \qquad \text{(A.}b\text{.13)}$$

(c) *Statistic* X_2

$$X_2 = \sum_{r=1}^{n}(x_r - y_r) + \sum_{r=1}^{n-1}\{(x_r - y_{r+1}) + (x_{r+1} - y_r)\} + \sum_{r=1}^{n-2}\{(x_r - y_{r+2}) + (x_{r+2} - y_r)\} \qquad \text{(A.}c\text{.1)}$$

As earlier, assuming $P_{\binom{AA}{AA}}(n, r)$, $P_{\binom{AA}{BA}}(n, r)$, etc., as the probabilities of X_2 having a value r when n is the size of the sequence and the first two observations of the sequences are $\binom{AA}{AA}$, $\binom{AA}{BA}$, etc., the following relations are obtained.

$$\left.\begin{aligned}
P_{\binom{AA}{AA}}(n, r) &= p^2P_{\binom{AA}{AA}}(n-1, r) + p^2P_{\binom{AA}{AB}}(n-1, r-1) + p^2P_{\binom{AB}{AA}}(n-1, r+1) + p^2P_{\binom{AB}{AB}}(n-1, r)\\
P_{\binom{AA}{AB}}(n, r) &= p^2P_{\binom{AA}{BA}}(n-1, r-1) + p^2P_{\binom{AA}{BB}}(n-1, r-2) + p^2P_{\binom{AB}{BA}}(n-1, r) + p^2P_{\binom{AB}{BB}}(n-1, r-1)\\
P_{\binom{AB}{AA}}(n, r) &= p^2P_{\binom{BA}{AA}}(n-1, r+1) + p^2P_{\binom{BA}{AB}}(n-1, r) + p^2P_{\binom{BB}{AA}}(n-1, r+2) + p^2P_{\binom{BB}{AB}}(n-1, r+1)\\
P_{\binom{AB}{AB}}(n, r) &= p^2P_{\binom{BA}{BA}}(n-1, r) + p^2P_{\binom{BA}{BB}}(n-1, r-1) + p^2P_{\binom{BB}{BA}}(n-1, r+1) + p^2P_{\binom{BB}{BB}}(n-1, r)
\end{aligned}\right\}$$

$$\left.\begin{aligned}
P_{\binom{AA}{BA}}(n, r) &= pqP_{\binom{AA}{AA}}(n-1, r-3) + pqP_{\binom{AA}{AB}}(n-1, r-4) + pqP_{\binom{AB}{AA}}(n-1, r-2) + pqP_{\binom{AB}{AB}}(n-1, r-3)\\
P_{\binom{AA}{BB}}(n, r) &= pqP_{\binom{AA}{BA}}(n-1, r-4) + pqP_{\binom{AA}{BB}}(n-1, r-5) + pqP_{\binom{AB}{BA}}(n-1, r-3) + pqP_{\binom{AB}{BB}}(n-1, r-4)\\
P_{\binom{AB}{BA}}(n, r) &= pqP_{\binom{BA}{AA}}(n-1, r-2) + pqP_{\binom{BA}{AB}}(n-1, r-3) + pqP_{\binom{BB}{AA}}(n-1, r-1) + pqP_{\binom{BB}{AB}}(n-1, r-2)\\
P_{\binom{AB}{BB}}(n, r) &= pqP_{\binom{BA}{BA}}(n-1, r-3) + pqP_{\binom{BA}{BB}}(n-1, r-4) + pqP_{\binom{BB}{BA}}(n-1, r-2) + pqP_{\binom{BB}{BB}}(n-1, r-3)
\end{aligned}\right\}$$

$$\left.\begin{aligned}
P_{\binom{BA}{AA}}(n,r) &= pqP_{\binom{AA}{AA}}(n-1,r+3) + pqP_{\binom{AA}{AB}}(n-1,r+2) + pqP_{\binom{AB}{AA}}(n-1,r+4) + pqP_{\binom{AB}{AB}}(n-1,r+3)\\
P_{\binom{BA}{AB}}(n,r) &= pqP_{\binom{AA}{BA}}(n-1,r+2) + pqP_{\binom{AA}{BB}}(n-1,r+1) + pqP_{\binom{AB}{BA}}(n-1,r+3) + pqP_{\binom{AB}{BB}}(n-1,r+2)\\
P_{\binom{BB}{AA}}(n,r) &= pqP_{\binom{BA}{AA}}(n-1,r+4) + pqP_{\binom{BA}{AB}}(n-1,r+3) + pqP_{\binom{BB}{AA}}(n-1,r+5) + pqP_{\binom{BB}{AB}}(n-1,r+4)\\
P_{\binom{BB}{AB}}(n,r) &= pqP_{\binom{BA}{BA}}(n-1,r+3) + pqP_{\binom{BA}{BB}}(n-1,r+2) + pqP_{\binom{BB}{BA}}(n-1,r+4) + pqP_{\binom{BB}{BB}}(n-1,r+3)
\end{aligned}\right\}$$

$$\left.\begin{aligned}
P_{\binom{BA}{BA}}(n,r) &= q^2P_{\binom{AA}{AA}}(n-1,r) + q^2P_{\binom{AA}{AB}}(n-1,r-1) + q^2P_{\binom{AB}{AA}}(n-1,r+1) + q^2P_{\binom{AB}{AB}}(n-1,r)\\
P_{\binom{BA}{BB}}(n,r) &= q^2P_{\binom{AA}{BA}}(n-1,r-1) + q^2P_{\binom{AA}{BB}}(n-1,r-2) + q^2P_{\binom{AB}{BA}}(n-1,r) + q^2P_{\binom{AB}{BB}}(n-1,r-1)\\
P_{\binom{BB}{BA}}(n,r) &= q^2P_{\binom{BA}{AA}}(n-1,r+1) + q^2P_{\binom{BA}{AB}}(n-1,r) + q^2P_{\binom{BB}{AA}}(n-1,r+2) + q^2P_{\binom{BB}{AB}}(n-1,r+1)\\
P_{\binom{BB}{BB}}(n,r) &= q^2P_{\binom{BA}{BA}}(n-1,r) + q^2P_{\binom{BA}{BB}}(n-1,r-1) + q^2P_{\binom{BB}{BA}}(n-1,r+1) + q^2P_{\binom{BB}{BB}}(n-1,r)
\end{aligned}\right\}$$

(A.c.2)

Expressing in terms of the P.G.F.'s and eliminating we get

$$\begin{vmatrix}
(E-p^2) & -p^2t & -p^2t^{-1} & -p^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E & 0 & 0 & -p^2t & -p^2t^2 & -p^2 & -p^2t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 & 0 & 0 & -p^2t^{-1} & -p^2 & -p^2t^{-2} & -p^2t^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p^2 & -p^2t & -p^2t^{-1} & -p^2 \\
-pqt^3 & -pqt^4 & -pqt^2 & -pqt^3 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -pqt^4 & (E-pqt^5) & -pqt^3 & -pqt^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E & 0 & -pqt^2 & -pqt^3 & -pqt & -pqt^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & -pqt^3 & -pqt^4 & -pqt^2 & -pqt^3 \\
-pqt^{-3} & -pqt^{-2} & -pqt^{-4} & -pqt^{-3} & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -pqt^{-2} & -pqt^{-1} & -ptq^{-3} & -pqt^{-2} & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -pqt^{-4} & -pqt^{-3} & (E-pqt^{-5}) & -pqt^{-4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & -pqt^{-3} & -pqt^{-2} & -pqt^{-4} & -pqt^{-3} \\
-q^2 & -q^2t & -q^2t^{-1} & -q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^2t & -q^2t^2 & -q^2 & -q^2t & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^2t^{-1} & -q^2 & -q^2t^{-2} & -q^2t^{-1} & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^2 & -q^2t & -q^2t^{-1} & (E-q^2)
\end{vmatrix} \phi(n, X_2) = 0 \qquad \text{(A. } c\text{. 3)}$$

Hence the recurrence relation connecting the P.G.F.'s, after evaluating the determinant, is

$$[E^{16} - E^{15}\{p^2 + q^2 + pqt^5 + pqt^{-5}\}]\,\phi(n, X_2) = 0 \qquad (A.c.4)$$

Putting $t = e^{\theta}$ in $(A.c.4)$ we get the difference equation of the M.G.F.'s.

$$[E^{16} - E^{15}\{p^2 + q^2 + pqe^{5\theta} + pqe^{-5\theta}\}]\,M(n, X_2) = 0 \qquad (A.c.5)$$

The only non-vanishing root of the characteristic equation of (A.c.5) is

$$\lambda = \{p^2 + q^2 + pqe^{5\theta} + pqe^{-5\theta}\},$$

which is unity for $\theta = 0$.

Thus

$$M(n, X_2) = A_1\,\lambda^n \qquad (A.c.6)$$

where A_1 is independent of n. Expressing A_1 in terms of λ and $M(3, X_2)$ the lowest M.G.F. obtainable from (A.c.6) we can write (A.c.6) as

$$M(n, X_2) = M(3, X_2)\,\lambda^{n-3}, \qquad (A.c.7)$$

$M(3, X_2)$ is as given below.

$$\begin{aligned} M(3, X_2) = {} & \{(p^6 + q^6) + 9p^2q^2(p^2 + q^2)\} \\ & + \{3pq(p^4 + q^4) + 9p^3q^3\}\{e^{3\theta} + e^{-3\theta}\} \\ & + \{3p^2q^2(p^2 + q^2)\}\{e^{6\theta} + e^{-6\theta}\} \\ & + p^3q^3\{e^{9\theta} + e^{-9\theta}\} \end{aligned} \qquad (A.c.8)$$

The rth cumulant of the distribution is

$$\kappa_r = \left[\frac{\partial^r}{\partial\theta^r}\log M(3, X_2)\right]_{\theta=0} + (n - 3)\left[\frac{\partial^r}{\partial\theta^r}\log\lambda\right]_{\theta=0} \qquad (A.c.9)$$

For $\theta = 0$

$$\begin{aligned} & M^{\text{I}}(3, X_2) = 0 & & \lambda^{\text{I}} = 0 \\ & M^{\text{II}}(3, X_2) = 54pq & & \lambda^{\text{II}} = 50pq \\ & M^{\text{III}}(3, X_2) = 0 & & \lambda^{\text{III}} = 0 \\ & M^{\text{IV}}(3, X_2) = 486pq(1 + 12pq) & & \lambda^{\text{IV}} = 1250pq \end{aligned} \qquad (A.c.10)$$

Using (A.c.10) the first four cumulants of X_2 are obtained and given below.

$$\left.\begin{aligned} & \kappa_1 = 0 & & \kappa_3 = 0 \\ & \kappa_2 = (25n - 48)\,2pq & & \kappa_4 = (625n - 1632)\,2pq\,(1 - 6pq) \end{aligned}\right\} \qquad (A.c.11)$$

(*d*) *Statistic* X_s

$$X_s = \sum_{r=1}^{n} (x_r - y_r) + \sum_{i=1}^{s} \sum_{r=1}^{n-i} \{x_r - y_{r+i}) + (x_{r+i} - y_r)\} \qquad (A.d.1)$$

The difference equation for X_s when $s \leqslant n/2$ (n even) and $\leqslant (n+1)/2$ (n odd), can be deduced from an examination of the same for $s = 0, 1,$ and 2. For X_0, X_1 and X_2 the characteristic equations are

$$X_0 :\!-\; [\alpha^4 - \alpha^3 (p^2 + q^2 + pqt + pqt^{-1})] = 0$$

$$X_1 :\!-\; [\alpha^4 - \alpha^3 (p^2 + q^2 + pqt^3 + pqt^{-3})] = 0$$

$$X_2 :\!-\; [\alpha^{16} - \alpha^{15} (p^2 + q^2 + pqt^5 + pqt^{-5})] = 0$$

It appears from the above that $\phi(n)$ for X_s will be

$$\phi(n, X_s) = A_1 [p^2 + q^2 + pqt^{(2s+1)} + pqt^{-(2s+1)}]^n \qquad (A.d.2)$$

or the M.G.F.

$$M(n, X_s) = A_2 [p^2 + q^2 + pqe^{(2s+1)\theta} + pqe^{-(2s+1)\theta}]^n \qquad (A.d.3)$$

Putting λ for the expression within the brace in (A.*d*.3) we get

$$M(n, X_s) = A_2 \lambda^n \qquad (A.d.4)$$

The approximate value of the rth cumulant of X_s will be

$$\kappa_r \sim n \left[\frac{\partial^r}{\partial \theta^r} \log \lambda \right]_{\theta=0} \qquad (A.d.5)$$

Thus

$$\left.\begin{array}{ll} \kappa_1 = 0 & \kappa_3 = 0 \\ \kappa_2 \sim n (2s+1)^2\, 2pq & \kappa_4 \sim n (2s+1)^4\, 2pq \end{array}\right\} \qquad (A.d.6)$$

But the recurrence relation (A.*d*.2) and (A.*d*.3) do not hold when $s > n/2$ (n even) and $> (n+1)/2$ (n odd).

The exact expressions for the expectation and variance of X_s can be worked out without using the relation (A.*d*.3).

We note $E(x_r - y_r) = 0$,

$V(x_r - y_r) = 2pq$,

$\text{cov}\,(x_r - y_r)\,(x_r - y_{r+i}) = pq$,

$\text{cov}\,(x_r - y_r)\,(x_{r+i} - y_r) = pq$.

The unconnected pairs like $(x_r - y_r)$, $(x_{r+i} - y_{r+i})$ are independent.

Then

$$E(X_s) = w_1 E(x_r - y_r) = 0$$

$$V(X_s) = w_1 V(x_r - y_r) + w_2 \operatorname{cov}(x_r - y_r)(x_r - y_{r+i}) + w_2 \operatorname{cov}(x_r - y_r)(x_{r+i} - y_r) \qquad (A.d.7)$$

$$= (w_1 + w_2)\, 2pq \text{ for } s \leqslant n/2 \ (n \text{ even}) \text{ and} \leqslant (n+1)/2$$

(n odd).

When $s > n/2$ (n even) and $> (n+1)/2$ (n odd)

$$V(X_s) = w_1 V(x_r - y_r) + w_3 \operatorname{cov}(x_r - y_r)(x_r - y_{r+i}) + w_3 \operatorname{cov}(x_r - y_r)(x_{r+i} - y_r)$$

$$= (w_1 + w_3)\, 2pq \qquad (A.d.8)$$

where

$w_1 =$ (number of differences in X_s) $= n(2s+1) - s(s+1)$

$w_2 = 2$ {number of connected pairs like $(x_r - y_r)$, $(x_r - y_{r+i})$ or $(x_r - y_r)$, $(x_{r+i} - y_r)$ when $s \leqslant n/2$ (n even) and $\leqslant (n+1)/2$ (n odd)}

$$= 2s\{n(2s+1) - \tfrac{1}{3}(5s^2 + 6s + 1)\}$$

$w_3 = 2$ {number of connected pairs like $(x_r - y_r)$, $(x_r - y_{r+i})$ or $(x_r - y_r)$, $(x_{r+i} - y_r)$ when $s > n/2$ (n even) and $> (n+1)/2$ (n odd)}

$$= \tfrac{1}{3}\{n(n-1)(6s - n + 2) - 2s(s^2 - 1)\}.$$

It may be noted that the coefficient of n in the variance of X_s when $s \leqslant n/2$ (n even) and $\leqslant (n+1)/2$ (n odd) is the same as in κ_2 (A.d.6).

When the probabilities for A and B are (p_1, q_1) and (p_2, q_2) for the first and second samples respectively, the first two cumulants reduce to

$$\kappa_1 = w_1(p_1 - p_2) \quad \text{and}$$

$$\kappa_2 = w_1 Z_1 + w_2 Z_2 \quad \text{for } s \leqslant n/2 \ (n \text{ even}) \text{ and} \leqslant (n+1)/2 \ (n \text{ odd})$$

$$= w_1 Z_1 + w_3 Z_2 \quad \text{for } s > n/2 \ (n \text{ even}) \text{ and} > (n+1)/2 \ n \text{ (odd)} \qquad (A.d.9)$$

where

$$Z_1 = (p_1 q_1 + p_2 q_2)$$

$$Z_2 = \{p_1 q_2 (p_1 + q_2) + q_1 p_2 (q_1 + p_2) - 2(p_1 - p_2)^2\}$$

$$= (p_1 q_1 + p_2 q_2).$$

B. Y *Statistics*

The *Y Statistics* are similar to the *X Statistics* with the algebraic differences $(x_r - y_{r+i})$ replaced by the absolute values $| x_r - y_{r+i} |$.

B. (*a*) *Statistic* Y_0

$$Y_0 = \sum_{r=1}^{n} | x_r - y_r | \qquad (\text{B}.a.1)$$

The P.G.F. and the cumulants of Y_0 are same as those of the Binomial distribution with $P = 2pq$ and $Q = 1 - P$. They are given below.

The P.G.F.

$$\phi(n, Y_0) = [p^2 + q^2 + 2pqt]^n \qquad (\text{B}.a.2)$$

The cumulants are

$$\begin{aligned} \kappa_1 &= 2\, npq, \\ \kappa_2 &= 2\, npq\, (1 - 2pq), \\ \kappa_3 &= 2\, npq\, (1 - 6pq + 8p^2q^2) \text{ and} \\ \kappa_4 &= 2\, npq\, (1 - 14pq + 48p^2q^2 - 48p^3q^3) \end{aligned} \qquad (\text{B}.a.3)$$

(*b*) *Statistic* Y_1

$$Y_1 = \sum_{r=1}^{n} | x_r - y_r | + \sum_{r=1}^{n-1} \{ |x_r - y_{r+1}| + |x_{r+1} - y_r| \} \qquad (\text{B}.b.1)$$

As in the case of X_1 the distribution of Y_1 can be discussed by deriving the difference equation connecting the P.G.F.'s or the M.G.F.'s.

Assuming $P_{\binom{A}{A}}(n, r)$, $P_{\binom{A}{B}}(n, r)$, etc., as in X_1 the following relations can be established.

$$\begin{aligned} P_{\binom{A}{A}}(n, r) &= p^2P_{\binom{A}{A}}(n-1, r) + p^2P_{\binom{A}{B}}(n-1, r-1) + p^2P_{\binom{B}{A}}(n-1, r-1) + p^2P_{\binom{B}{B}}(n-1, r-2) \\ P_{\binom{A}{B}}(n, r) &= pqP_{\binom{A}{A}}(n-1, r-2) + pqP_{\binom{A}{B}}(n-1, r-3) + pqP_{\binom{B}{A}}(n-1, r-1) + pqP_{\binom{B}{B}}(n-1, r-2) \\ P_{\binom{B}{A}}(n, r) &= pqP_{\binom{A}{A}}(n-1, r-2) + pqP_{\binom{A}{B}}(n-1, r-1) + pqP_{\binom{B}{A}}(n-1, r-3) + pqP_{\binom{B}{B}}(n-1, r-2) \\ P_{\binom{B}{B}}(n, r) &= q^2P_{\binom{A}{A}}(n-1, r-2) + q^2P_{\binom{A}{B}}(n-1, r-1) + q^2P_{\binom{B}{A}}(n-1, r-1) + q^2P_{\binom{B}{B}}(n-1, r) \end{aligned}$$

(B.*b*.2)

Reducing in terms of P.G.F.'s. and eliminating

$$\begin{vmatrix} (E-p^2) & -p^2t & -p^2t & -p^2t^2 \\ -pqt^2 & (E-pqt^3) & -pqt & -pqt^2 \\ -pqt^2 & -pqt & (E-pqt^3) & -pqt^2 \\ -q^2t^2 & -q^2t & -q^2t & (E-q^2) \end{vmatrix} \phi(n, Y_1) = 0 \quad (\text{B}.b.3)$$

Evaluation of the determinant reduces the difference equation to

$$[E^4 - E^3\{p^2 + q^2 + 2pqt^3\} + E^2\{p^2q^2(1-t^4)(1-t^2)\} + E\{p^4q^2t^2(1-t^2)^2 + 2p^3q^3t^3(1-t^2)^2 + p^2q^4t^2(1-t^2)^2\} - p^4q^4t^2(1-t^2)^4]\,\phi(n, Y_1) = 0 \quad (\text{B}.b.4)$$

For $n = 1$, equation (B.b.4) will be

$$\phi(5, Y_1) - \phi(4, Y_1)\{p^2 + q^2 + 2pqt^3\} + \phi(3, Y_1)\{p^2q^2(1-t^4)(1-t^2)\} + \phi(2, Y_1)\{p^4q^2t^2(1-t^2)^2 + 2p^3q^3t^3(1-t^2)^2 + p^2q^4t^2(1-t^2)^2\} - \phi(1, Y_1)\{p^4q^4t^2(1-t^2)^4\} = 0 \quad (\text{B}.b.5)$$

With the substitution of $\phi(5, Y_1)$, $\phi(4, Y_1)$, $\phi(3, Y_1)$, $\phi(2, Y_1)$ and $\phi(1, Y_1)$ obtained by direct enumeration, the left side of equation (B.b.5) vanishes. Putting $t = e^\theta$ in (B.b.4) the difference equation satisfied by the M.G.F.'s is obtained.

$$[E^4 - E^3\{p^2+q^2+2pqe^{3\theta}\} + E^2\{p^2q^2(1-e^{4\theta})(1-e^{2\theta})\} + E\{p^4q^2e^{2\theta}(1-e^{2\theta})^2 + 2p^3q^3e^{3\theta}(1-e^{2\theta})^2 + p^2q^4e^{2\theta}(1-e^{2\theta})^2\} - p^4q^4e^{2\theta}(1-e^{2\theta})^4]\,M(n, Y_1) = 0 \quad (\text{B}.b.6)$$

For $\theta = 0$ the characteristic equation of (B.b.6) comes out to be

$$\alpha^3(\alpha - 1) = 0 \quad (\text{B}.b.7)$$

That is, in this case three roots of the characteristic equation of (B.b.6) vanish and the fourth root is unity. This property of the difference equation makes the calculation of the cumulants much easier. Following the methods developed by one of the authors [1954] the first four cumulants of Y_1 are evaluated and given below.

$$\kappa_1 = 6npq - 4pq$$

$$\kappa_2 = 6npq(3 - 10pq) - 4pq(5 - 18pq),$$

$$\kappa_3 = 6npq(9 - 94pq + 232p^2q^2) - 4pq(19 - 216pq + 560p^2q^2),$$

$$\kappa_4 = 6npq(27 - 682pq + 4464p^2q^2 - 8624p^3q^3) - 4pq(65 - 1806pq + 12{,}528p^2q^2 - 25{,}296p^3q^3).$$

(*c*) *Statistic* Y_2

$$Y_2 = \sum_{r=1}^{n} | x_r - y_r | + \sum_{i=1}^{2} \sum_{r=1}^{n-i} \{| x_r - y_{r+i} | + | x_{r+i} - y_r |\} \tag{B.c.1}$$

Assuming $P_{\binom{AA}{AA}}(n, r)$, $P_{\binom{AA}{BA}}(n, r)$, etc., as in X_2 we can establish the following relations:

$$P_{\binom{AA}{AA}}(n, r) = p^2 P_{\binom{AA}{AA}}(n-1, r) + p^2 P_{\binom{AA}{AB}}(n-1, r-1) + p^2 P_{\binom{AB}{AA}}(n-1, r-1) + p^2 P_{\binom{AB}{AB}}(n-1, r-2)$$

$$P_{\binom{AA}{AB}}(n, r) = p^2 P_{\binom{AA}{BA}}(n-1, r-1) + p^2 P_{\binom{AA}{BB}}(n-1, r-2) + p^2 P_{\binom{AB}{BA}}(n-1, r-2) + p^2 P_{\binom{AB}{BB}}(n-1, r-3)$$

$$P_{\binom{AB}{AA}}(n, r) = p^2 P_{\binom{BA}{AA}}(n-1, r-1) + p^2 P_{\binom{BA}{AB}}(n-1, r-2) + p^2 P_{\binom{BB}{AA}}(n-1, r-2) + p^2 P_{\binom{BB}{AB}}(n-1, r-3)$$

$$P_{\binom{AB}{AB}}(n, r) = p^2 P_{\binom{BA}{BA}}(n-1, r-2) + p^2 P_{\binom{BA}{BB}}(n-1, r-3) + p^2 P_{\binom{BB}{BA}}(n-1, r-3) + p^2 P_{\binom{BB}{BB}}(n-1, r-4)$$

$$P_{\binom{AA}{BA}}(n, r) = pq P_{\binom{AA}{AA}}(n-1, r-3) + pq P_{\binom{AA}{AB}}(n-1, r-4) + pq P_{\binom{AB}{AA}}(n-1, r-2) + pq P_{\binom{AB}{AB}}(n-1, r-3)$$

$$P_{\binom{AA}{BB}}(n, r) = pq P_{\binom{AA}{BA}}(n-1, r-4) + pq P_{\binom{AA}{BB}}(n-1, r-5) + pq P_{\binom{AB}{BA}}(n-1, r-3) + pq P_{\binom{AB}{BB}}(n-1, r-4)$$

$$P_{\binom{AB}{BA}}(n, r) = pq P_{\binom{BA}{AA}}(n-1, r-2) + pq P_{\binom{BA}{AB}}(n-1, r-3) + pq P_{\binom{BB}{AA}}(n-1, r-1) + pq P_{\binom{BB}{AB}}(n-1, r-2)$$

$$P_{\binom{AB}{BB}}(n, r) = pq P_{\binom{BA}{BA}}(n-1, r-3) + pq P_{\binom{BA}{BB}}(n-1, r-4) + pq P_{\binom{BB}{BA}}(n-1, r-2) + pq P_{\binom{BB}{BB}}(n-1, r-3)$$

$$P_{\binom{BA}{AA}}(n, r) = pqP_{\binom{AA}{AA}}(n-1, r-3) + pqP_{\binom{AA}{AB}}(n-1, r-2) + pqP_{\binom{AB}{AA}}(n-1, r-4) + pqP_{\binom{AB}{AB}}(n-1, r-3)$$

$$P_{\binom{BA}{AB}}(n, r) = pqP_{\binom{AA}{BA}}(n-1, r-2) + pqP_{\binom{AA}{BB}}(n-1, r-1) + pqP_{\binom{AB}{BA}}(n-1, r-3) + pqP_{\binom{AB}{BB}}(n-1, r-2)$$

$$P_{\binom{BB}{AA}}(n, r) = pqP_{\binom{BA}{AA}}(n-1, r-4) + pqP_{\binom{BA}{AB}}(n-1, r-3) + pqP_{\binom{BB}{AA}}(n-1, r-5) + pqP_{\binom{BB}{AB}}(n-1, r-4)$$

$$P_{\binom{BB}{AB}}(n, r) = pqP_{\binom{BA}{BA}}(n-1, r-3) + pqP_{\binom{BA}{BB}}(n-1, r-2) + pqP_{\binom{BB}{BA}}(n-1, r-4) + pqP_{\binom{BB}{BB}}(n-1, r-3)$$

$$P_{\binom{BA}{BA}}(n, r) = q^2P_{\binom{AA}{AA}}(n-1, r-4) + q^2P_{\binom{AA}{AB}}(n-1, r-3) + q^2P_{\binom{AB}{AA}}(n-1, r-3) + q^2P_{\binom{AB}{AB}}(n-1, r-2)$$

$$P_{\binom{BA}{BB}}(n, r) = q^2P_{\binom{AA}{BA}}(n-1, r-3) + q^2P_{\binom{AA}{BB}}(n-1, r-2) + q^2P_{\binom{AB}{BA}}(n-1, r-2) + q^2P_{\binom{AB}{BB}}(n-1, r-1)$$

$$P_{\binom{BB}{BA}}(n, r) = q^2P_{\binom{BA}{AA}}(n-1, r-3) + q^2P_{\binom{BA}{AB}}(n-1, r-2) + q^2P_{\binom{BB}{AA}}(n-1, r-2) + q^2P_{\binom{BB}{AB}}(n-1, r-1)$$

$$P_{\binom{BB}{BB}}(n, r) = q^2P_{\binom{BA}{BA}}(n-1, r-2) + q^2P_{\binom{BA}{BB}}(n-1, r-1) + q^2P_{\binom{BB}{BA}}(n-1, r-1) + q^2P_{\binom{BB}{BB}}(n-1, r)$$

Expressing these equations in terms of P.G.F.'s of Y_2 and eliminating

$$\begin{vmatrix}
(E-p^2) & -p^2t & -p^2t & -p^2t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & E & 0 & 0 & -p^2t & -p^2t^2 & -p^2t^2 & -p^2t^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & E & 0 & 0 & 0 & 0 & 0 & -p^2t & -p^2t^2 & -p^2t^2 & -p^2t^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p^2t^2 & -p^2t^3 & -p^2t^3 & -p^2t^4 \\
-pqt^3 & -pqt^4 & -pqt^2 & -pqt^3 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -pqt^4 & (E-pqt^5) & -pqt^3 & -pqt^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & E & 0 & -pqt^2 & -pqt^3 & -pqt & -pqt^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & -pqt^3 & -pqt^4 & -pqt^2 & -pqt^3 \\
-pqt^3 & -pqt^2 & -pqt^4 & -pqt^3 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -pqt^2 & -pqt & -pqt^3 & -pqt^2 & 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -pqt^4 & -pqt^3 & (E-pqt^5) & -pqt^4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & -pqt^3 & -pqt^2 & -pqt^4 & -pqt^3 \\
-q^2t^4 & -q^2t^3 & -q^2t^3 & -q^2t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^2t^3 & -q^2t^2 & -q^2t^2 & -q^2t & 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^2t^3 & -q^2t^2 & -q^2t^2 & -q^2t & 0 & 0 & E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^2t^2 & -q^2t & -q^2t & (E-q^2)
\end{vmatrix} \phi(n, Y_2) = 0$$

It is rather difficult to obtain the actual form of the difference equation for Y_2 as the solution of the above determinant is laborious. The first two cumulants of Y_2 are

$$\kappa_1 = 10npq - 12pq$$
$$\kappa_2 = 10npq(5 - 18pq) - 4pq(25 - 94pq) \qquad \text{(B.}c\text{.4)}$$

For $t = 1$, the above determinant reduces to $E^{15}(E - 1)$. Therefore when $t = 1$, the characteristic equation is $\alpha^{15}(\alpha - 1) = 0$. That is, all the roots vanish except one which is unity. Therefore following the arguments given in a previous paper (1954) it can be seen that all the cumulants of this distribution are linear functions in n.

(*d*) *Statistic* Y_s

$$Y_s = \sum_{r=1}^{n} |x_r - y_r| + \sum_{i=1}^{s} \sum_{r=1}^{n-i} \{|x_r - y_{r+i}| + |x_{r+i} - y_r|\} \qquad \text{(B.}d\text{.1)}$$

It is not possible to deduce the difference equation of Y_s as in the case of X_s. The exact expressions for the expectation and variance of Y_s are worked out below:—

$$E|x_r - y_r| = 2pq$$
$$V|x_r - y_r| = 2pq(1 - 2pq)$$
$$\text{Cov}\{|x_r - y_r|\ |x_r - y_{r+i}|\} = pq(1 - 4pq)$$
$$\text{Cov}\{|x_r - y_r|\ |x_{r+i} - y_r|\} = pq(1 - 4pq)$$

and unconnected pairs are independent.

So

$$E(Y_s) = w_1 E|x_r - y_r| = w_1 2pq,$$
$$V(Y_s) = w_1 V(|x_r - y_r|) \qquad \text{(B.}d\text{.2)}$$
$$+ w_2 \text{cov}(|x_r - y_r|\ |x_r - y_{r+i}|)$$
$$+ w_2 \text{cov}(|x_r - y_r|\ |x_{r+i} - y_r|)$$
$$= w_1 2pq + w_2 2pq(1 - 4pq)$$
for $s \leqslant n/2$ (n even) and $\leqslant (n+1)/2$ (n odd).

When $s > n/2$ (n even) and $> (n+1)/2$ (n odd)

$$V(Y_s) = w_1 2pq + w_3 2pq(1 - 4pq)$$

where w_1, w_2 and w_3 are as given in (A.d.7).

When the probabilities for A and B are (p_1, q_1) and (p_2, q_2) for the first and second samples respectively the expectation and variance of Y_s will reduce to

$$E(Y_s) = w_1(p_1q_2 + q_1p_2)$$

$$V(Y_s) = w_1Z_3 + w_2Z_4, \quad \text{for } s \leqslant n/2 \ (n \text{ even}) \text{ and} \leqslant (n+1)/2 \ (n \text{ odd}),$$

$$= w_1Z_3 + w_3Z_4 \quad \text{for } s > n/2 \ (n \text{ even}) \text{ and} > (n+1)/2 \ (n \text{ odd}).$$

where

$$Z_3 = (p_1q_2+q_1p_2)\{1 - (p_1q_2 + q_1p_2)\},$$

$$Z_4 = \{p_1q_2(p_1+q_2) + q_1p_2(q_1+p_2) - 2(p_1q_2+q_1p_2)^2\} \qquad (B.d.3)$$

3. DISTRIBUTIONS OF X AND Y STATISTICS

(a) *Small Sample Exact Distributions*.—From the discussions regarding the power of the tests in Section 4 it would be seen that for all practical purposes it seems to be sufficient if the discussions are confined to the exact distributions of *Statistics* X_0 and Y_1 when n ranges from 1 to 6. The exact distributions of the *Statistics* X_0 and Y_1 for different values of n ($1 - 6$ for X_0 and $2 - 6$ for Y_1) have been obtained for values of $p = 0{\cdot}1\ (0{\cdot}1)\ 0{\cdot}5$ and given in Tables I and II. These tables will enable us to calculate the probability that an observed value of X_0 or Y_1 exceeds (or is less than) a certain quantity r. The probability generating functions for X_0 and Y_1 are given in the Appendix.

For larger n the exact probabilities can be calculated by making use of the recurrence relations for X_0 and Y_1 (A.a.2 and B.b.4) and the probabilities given in the above tables.

For X_0 the recurrence relations can be written as

$$\phi(n, X_0) = \phi(n-1, X_0)[p^2 + q^2 + pqt + pqt^{-1}]$$

Equating the coefficients of t^r on both the sides we get

$$P(n, r)_{X_0} = P(n-1, r)_{X_0}(p^2+q^2) + \{P(n-1, r-1)_{X_0} + P(n-1, r+1)_{X_0}\}pq \qquad (3.a.1)$$

where $P(n, r)_{X_0}$ is the probability of X_0 having a value r when the size of the sequences is n.

Similarly for Y_1 the recurrence relation is

$$\begin{aligned}\phi(n+4, Y_1) = {} & \phi(n+3, Y_1)\{p^2 + q^2 + 2pqt^3\} \\ & - \phi(n+2, Y_1)\{p^2q^2(1-t^4)(1-t^2)\} \\ & - \phi(n+1, Y_1)\{p^4q^2t^2(1-t^2)^2 \\ & + 2p^3q^3t^3(1-t^2)^2 + p^2q^4t^2(1-{}^2)^2\} \\ & + \phi(n, Y_1)\{p^4q^4t^2)(1-t^2)^4\}.\end{aligned}$$

TABLE I

Exact probability distributions of Statistic X_0

$p = \cdot 1;\ q = \cdot 9.$

n	X_0												
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
1	..	..	..	..	..	·0900	·8200	·0900	..	..	..	..	..
2	..	..	..	..	·0081	·1476	·6886	·1476	·0081	..	..	..	..
3	..	..	..	·0007	·0199	·1837	·5912	·1837	·0199	·007	..	..	..
4	..	..	0066*	·0024	·0329	·2057	·5179	·2057	·0329	·0024	0066*	..	..
5	..	0006*	·0003	·0049	·0457	·2182	·4617	·2182	·0457	·0049	·0003	0006*	..
6	0053†	0029*	·0007	·0082	·0575	·2246	·4179	·2246	·0575	·0082	·0007	0029*	0053†

* Prefix ·00 to each entry.
† Prefix ·0000 to each entry.

TABLE I (*Contd.*)

$p = \cdot 2;\ q = \cdot 8.$

n	X_0												
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
1	..	..	..	..	..	·1600	·6800	·1600	..	..	..	..	..
2	..	..	..	..	·0256	·2176	·5136	·2176	·0256	..	..	..	..
3	..	..	..	·0041	·0522	·2342	·4189	·2342	·0522	·0041	..	..	..
4	..	..	·0007	·0111	·0736	·2347	·3598	·2347	·0736	·0111	·0007	..	..
5	..	0010*	·0022	·0195	·0894	·2289	·3198	·2289	·0894	·0195	·0022	0010*	..
6	1678†	0428*	·0046	·0279	·1005	·2211	·2907	·2211	·1005	·0279	·0046	0428*	1678†

* Prefix ·00 to each entry.
† ,, ·0000 ,, ,,

TABLE I—*Contd.*

$p = \cdot 3;\ q = \cdot 7.$

n	X_0												
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
1	..	..	..	..	..	·2100	·5800	·2100	..	..	..	..	..
2	..	..	..	..	·0441	·2436	·4246	·2436	·0441	..	..	..	..
3	..	..	..	·0093	·0767	·2397	·3486	·2397	·0767	·0093	..	..	..
4	..	..	·0019	·0215	·0968	·2284	·3028	·2284	·0968	·0215	·0019	..	..
5	..	·0004	·0056	·0332	·1086	·2164	·2716	·2164	·1086	·0332	·0056	·0004	..
6	0086*	·0014	·0103	·0432	·1154	·2053	·2484	·2053	·1154	·0432	·0103	·0014	0086*

* Prefix ·00 to each entry

TABLE I—*Contd.*

$p = \cdot 4;\ q = \cdot 6$

n	X_0												
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
1	..	..	..	..	..	·2400	·5200	·2400	..	..	..	..	..
2	..	..	..	..	·0576	·2496	·3856	·2496	·0576	..	..	..	..
3	..	..	..	0·0138	·0899	·2362	·3203	·2362	·0899	·0138	..	..	..
4	..	..	·0033	0·0288	·1067	·2212	·2799	·2212	·1067	·0288	·0033	..	..
5	..	·0008	·0086	0·0414	·1155	·2078	·2518	·2078	·1155	·0414	·0086	·0008	..
6	·0002	·0025	·0146	0·0513	·1199	·1962	·2307	·1962	·1199	·0513	·0146	·0025	·0002

TABLE I—*Contd.*

$p = \cdot 5;\ q = \cdot 5.$

n	X_0												
	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
1	..	..	..	..	..	·2500	·5000	·2500	..	..	..	..	..
2	..	..	..	..	·0625	·2500	·3750	·2500	·0625	..	..	..	..
3	..	..	..	·0156	·0938	·2344	·3125	·2344	·0938	·0156	..	..	..
4	..	..	·0039	·0312	·1094	·2188	·2734	·2188	·1094	·0312	·0039	..	..
5	..	·0010	·0098	·0439	·1172	·2051	·2460	·2051	·1172	·0439	·0098	·0010	..
6	·0002	·0029	·0161	·0537	·1208	·1934	·2256	·1934	·1208	·0537	·0161	·0029	·0002

TABLE II

Exact probability distributions of statistic Y_1

$p = \cdot 1;\ q = \cdot 9$

n	Y_1																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	·6562	0	·3276	0	·0162	..	..	..	..	..	..	..	..	..	..	..	..
3	·5314	0	·2495	·1491	·0390	·0295	0	·0015	..	..	..	..	..	..	..	..	..
4	·4305	0	·2021	·2150	·0502	·0749	·0191	·0053	·0028	0	·0001	..	..	..	..	..	..
5	·3487	0	·1637	·2516	·0535	·1002	·0526	·0149	·0119	·0020	·0007	·0003	0	0012*	..	..	..
6	·2824	0	·1326	·2666	·0539	·1129	·0863	·0258	·0259	·0092	·0026	·0016	·0002	0082*	0025*	0	0001*

* Prefix ·00 to each entry.

TABLE II—*Contd.*

$p = \cdot 2;\ q = \cdot 8.$

n	Y_1																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	·4112	0	·5376	0	·0512	..	..	..	..	..	..	..	..	..	..	..	..
3	·2622	0	·2980	·2258	·1198	·0860	0	·0082	..	..	..	..	..	..	..	..	..
4	·1678	0	·1902	·2211	·1229	·1866	·0676	·0275	·0151	0	·0013	..	..	..	..	..	..
5	·1074	0	·1217	·1951	·0980	·1928	·1340	·0721	·0560	·0139	·0062	·0026	0	·0002	..	..	..
6	·0687	0	·0779	·1592	·0759	·1689	·1577	·0982	·1023	·0506	·0227	·0134	·0028	·0013	·0004	0	0034*

* Prefix ·00 to each entry.

TABLE II—*Contd.*

$p = \cdot 3;\ q = \cdot 7.$

n	Y_1																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	·2482	0	·6636	0	·0882	..	..	..	..	..	..	..	..	..	..	..	..
3	·1184	0	·2596	·2621	·2020	·1394	0	·0185	..	..	..	..	..	..	..	..	..
4	·0577	0	·1252	·1647	·1676	·2640	·1252	·0585	·0332	0	·0039	..	..	..	..	..	..
5	·0283	0	·0612	·1044	·0962	·2047	·1891	·1445	·1106	·0357	·0168	·0078	0	·0008	..	..	..
6	·0138	0	·0300	·0630	·0552	·1329	·1565	·1540	·1720	·1097	·0610	·0355	·0098	·0045	·0018	0	·0002

TABLE II—*Contd.*

$p = \cdot 4;\ q = \cdot 6.$

n	Y_1																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	·1552	0	·7296	0	·1152	..	..	..	..	..	..	..	..	..	..	..	..
3	·0508	0	·2089	·2772	·2604	·1751	0	·0276	..	..	..	..	..	..	..	..	..
4	·0175	0	·0732	·1132	·1843	·3019	·1705	·0840	·0487	0	·0066	..	..	..	..	..	..
5	·0062	0	·0260	·0486	·0763	·1801	·2131	·1991	·1513	·0574	·0271	·0133	0	·0016	..	..	..
6	·0022	0	·0093	·0203	·0314	·0840	·1274	·1735	·2093	·1578	·0978	·0562	·0185	·0082	·0036	0	·0004

TABLE II—*Contd.*

$p = \cdot 5;\ q = \cdot 5.$

n	Y_1																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	·1250	0	·7500	0	·1250	..	..	..	..	..	..	..	..	..	..	..	..
3	·0313	0	·1875	·2812	·2812	·1875	0	·0313	..	..	..	..	..	..	..	..	..
4	·0078	0	·0547	·0937	·1875	·3125	·1875	·0937	·0547	0	·0078	..	..	..	..	..	..
5	·0020	0	·0156	·0313	·0664	·1660	·2187	·2187	·1660	·0664	·0313	·0156	0	·0020	..	..	..
6	·0005	0	·0044	·0098	·0225	·0644	·1128	·1758	·2197	·1758	·1128	·0644	·0225	·0098	·0044	0	·0005

Equating the coefficients of t^r on both the sides the following relation is obtained:—

$$\begin{aligned} P(n+4, r)_{Y_1} = {} & P(n+3, r)_{Y_1}(p^2+q^2) + P(n+3, r-3)_{Y_1} 2pq \\ & - \{P(n+2, r)_{Y_1} - P(n+2, r-2)_{Y_1} \\ & - P(n+2, r-4)_{Y_1} + P(n+2, r-6)_{Y_1}\} p^2q^2 \\ & - \{P(n+1, r-2)_{Y_1} + P(n+1, r-6)_{Y_1} \\ & - 2P(n+1, r-4)_{Y_1}\} p^2q^2(p^2+q^2) \\ & - \{P(n+1, r-3)_{Y_1} + P(n+1, r-7)_{Y_1} \\ & - 2P(n+1, r-5)_{Y_1}\} 2p^3q^3 \\ & + \{P(n, r-2)_{Y_1} - 4P(n, r-4)_{Y_1} \\ & + 6P(n, r-6)_{Y_1} - 4P(n, r-8)_{Y_1} \\ & + P(n, r-10)_{Y_1}\} p^4q^4 \end{aligned} \quad (3.a.2)$$

From the relations (3.*a*.1) and (3.*a*.2) the probabilities $P(n, r)_{X^0}$ and $P(n+4, r)_{Y_1}$ for any $n\ (> 6)$ and r can be obtained by substituting in the right-hand side in succession the values taken from the Tables I and II. It is proposed to prepare similar tables for higher values of n and s.

(*b*) *Asymptotic Distributions.*—It has already been shown in Section 2 that the first four cumulants of X_s and Y_s for $s = 0$, 1 and 2 are linear functions in n. In fact so long as s is finite it can be easily established that the cumulants of any order will be linear function in n and

$$\gamma_t = \frac{\kappa_{t+2}}{\kappa_2^{\frac{t+2}{2}}} = 0\left(\frac{1}{n^{\frac{t}{2}}}\right) \rightarrow 0$$

as $n \rightarrow \infty$ and therefore the distributions tend to the normal form so long as p is finite.

For larger values of s and n when $s = nf$ where f is a finite fraction it would be noted that the first two cumulants of X_s and Y_s are of degree 2 and 3 respectively in n. From the ideas developed in a previous paper (1952) it can be noted that the rth cumulant will be of degree $(r+1)$ in n. This can be seen from the fact that the rth cumulant is n times the expectation for r connected differences like $(x_l - y_l)$, $(x_l - y_{l+1})$, $(x_l - y_{l-1})$, etc. The highest degree of s will correspond to the expectation of these differences wherein the connected differences will involve maximum number of observations. In the particular type of distributions discussed here the number of ways of obtaining r connected differences wherein a maximum of $(r + 1)$ observations are involved is $(2s + 1)^{[r]}$. This is obtained by associating any observation of the first or the second sequence with r observation selected from the $(2s + 1)$ nearest observations of the other sequence. Thus the degree of n in the rth cumulant will be $(r + 1)$. Hence

$$\gamma_t = \frac{\kappa_{t+2}}{\kappa_2^{\frac{t+2}{2}}} = 0\left(\frac{1}{n^{\frac{t}{2}}}\right) \to 0$$

as $n \to \infty$.

It follows that the asymptotic distributions of X_s and Y_s for any value of s will take the normal form. It will further be noted that X statistics are always symmetrical. But Y_s is symmetrical when $p = q = \frac{1}{2}$. When p is very small the distribution of Y_0 tends to the Poisson form.

4. Power of X and Y Statistics for Comparing Two Binomial Sequences

In view of the fact that the distributions of the *Statistics* tend to the normal form for large values of n they can be used for comparing two binomial sequences. This may be done by calculating the standardized deviates of the observed values of the *Statistics*. It would be worth while to examine how the tests are affected when s increases. One of the ways of doing this is to calculate the power of the tests for different values of s and this has been done for X_s and Y_s for $n = 30$ and H_0: — $p_1 = p_2 = 0\cdot5$ and alternatives $p_1 = 0\cdot5$ and $p_2 = 0\cdot1$, $(0\cdot1)$ $0\cdot4$ and given in the Tables III and IV. These tables show that the power of X_s is maximum for $s = 0$. As regards Y_s, its power increases with s and attains a maximum at $s = (n - 1)$. The increase in power

TABLE III

Power of X_s *for testing the difference between two binomial samples*

$H_0 : p_1 = p_2 = \cdot5,\ q_1 = q_2 = \cdot5,\ n = 30,\ \alpha = 0\cdot05$

s	Alternatives p_2			
	·1	·2	·3	·4
0	·91681	·66089	33459	·11851
1	·91308	·65266	33211	·11801
5	·90847	·64363	32671	·11655
10	·90427	·63723	32284	·11560

TABLE IV

Power of Y_s for testing the difference between two binomial samples

$H_0 : p_1 = p_2 = \cdot 5,\ q_1 = q_2 = \cdot 5,\ n = 30,\ \alpha = 0 \cdot 05$

s	Alternatives p_2			
	·1	·2	·3	·4
0	·05000	·05000	·05000	·05000
1	·19164	·13310	·08692	·05908
5	·45658	·34658	·21354	·09412
10	·57000	·46306	·30994	·13066
15	·61972	·51844	·36338	·15640
20	·64298	·54514	·39092	·17152
25	·65438	·55846	·40508	·18020
29	·65766	·56226	·40920	·18234

is small after a certain value of s ranging from 5 to 20 depending upon the value of the alternative p_2. When n is small and ranges from 2–6, Y_1 and X_0 can be used for the purpose of comparison on the basis of the exact distributions given in Tables I and II.

Comparing Tables III and IV it would be noted that the power of Y *Statistics* for $s > 15$ is slightly more than that of X_0. It appears therefore that the Y *Statistics* ($s > 15$) may be more powerful than X_0. Detailed investigations on the relative efficiencies are in progress and will be published separately.

5. SUMMARY

The distributions of a number of *Statistics* arising from two binomial sequences

$\{x\}$— $x_1, x_2, x_3, \ldots, x_n$

$\{y)$— $y_1, y_2, y_3, \ldots, y_n$

have been discussed in this paper. The *Statistics* considered are the following:—

$$X_s = \sum_{r=1}^{n} (x_r - y_r) + \sum_{i=1}^{s} \sum_{r=1}^{n-i} \{(x_r - y_{r+i}) + (x_{r+i} - y_r)\}$$

$$Y_s = \sum_{r=1}^{n} | x_r - y_r | + \sum_{i=1}^{s} \sum_{r=1}^{n-i} \{| x_r - y_{r+i} | + | x_{r+i} - y_r |\}$$

$$(s = 0, 1, 2, \ldots, n - 1).$$

It has been shown that all these distributions tend to the normal form as $n \to \infty$. The standardized deviates of these Statistics can be used for testing the significance of two binomial sequences. The exact distribution of X_0 (n taking values 1 to 6) and Y_1 (n taking values 2 to 6) have been given. Distributions for $n > 6$ can be evaluated by using the recurrence relations given in the paper.

The power of the Statistics X_s and Y_s for different values of s for H_0: $p_1 = p_2 = 0 \cdot 5$ have been given for $n = 30$ and alternatives $p_1 = 0 \cdot 5$ and $p_2 = 0 \cdot 1$ $(0 \cdot 1)$ $0 \cdot 4$.

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APPENDIX

The probability generating functions for X_0 $(n = 1 - 6)$ and Y_1 $(n = 2 - 6)$ are given below:—

$$\phi(1, X_0) = (p^2 + q^2) + pq\,(t + t^{-1})$$

$$\begin{aligned}\phi(2, X_0) = &\{p^4 + q^4 + 4\,p^2q^2\} \\ &+ 2pq\,(p^2+q^2)\,(t+t^{-1}) \\ &+ p^2q^2\,(t^2+t^{-2})\end{aligned}$$

$$\begin{aligned}\phi(3, X_0) = &\{(p^6+q^6) + 9p^2q^2\,(p^2+q^2)\} \\ &+ \{3pq\,(p^4+q^4) + 9p^3q^3\}\,(t+t^{-1}) \\ &+ \{3p^2q^2\,(p^2+q^2)\}\,(t^2+t^{-2}) \\ &+ p^3q^3\,(t^3+t^{-3})\end{aligned}$$

$$\begin{aligned}\phi(4, X_0) = &\{(p^8+q^8) + 16p^2q^2\,(p^4+q^4) + 36p^4q^4\} \\ &+ \{4pq\,(p^6+q^6) + 24p^3q^3\,(p^2+q^2)\}\,(t+t^{-1}) \\ &+ \{6p^2q^2\,(p^4+q^4) + 16p^4q^4\}\,(t^2+t^{-2}) \\ &+ \{4p^3q^3\,(p^2+q^2)\}\,(t^3+t^{-3}) \\ &+ p^4q^4\,(t^4+t^{-4}).\end{aligned}$$

$$\begin{aligned}\phi(5, X_0) = &\{(p^{10}+q^{10}) + 25p^2q^2\,(p^6+q^6) + 100p^4q^4\,(p^2+q^2)\} \\ &+ \{5pq\,(p^8+q^8) + 50p^3q^3\,(p^4+q^4) + 100p^5q^5\} \\ &\quad \times (t + t^{-1}) \\ &+ \{10p^2q^2\,(p^6+q^6) + 50p^4q^4\,(p^2+q^2)\}\,(t^2+t^{-2}) \\ &+ \{10p^3q^3\,(p^4+q^4) + 25p^5q^5\}\,(t^3+t^{-3}) \\ &+ \{5p^4q^4\,(p^2+q^2)\}\,(t^4+t^{-4}) \\ &+ p^5q^5\,(t^5+t^{-5}).\end{aligned}$$

$$\begin{aligned}\phi(6, X_0) = &\{p^{12}+q^{12}) + 36p^2q^2\,(p^8+q^8) + 225p^4q^4(p^4+q^4) + 400p^6q^6\} \\ &+ \{6pq\,(p^{10}+q^{10}) + 90p^3q^3\,(p^6+q^6) \\ &+ 300p^5q^5\,(p^2+q^2)\}\,(t+t^{-1})\end{aligned}$$

$$+ \{15p^2q^2(p^8+q^8) + 120p^4q^4(p^4+q^4)$$
$$+ 225p^6q^6\}(t^2+t^{-2})$$
$$+ \{20p^3q^3(p^6+q^6) + 90p^5q^5(p^2+q^2)\}(t^3+t^{-3})$$
$$+ \{15p^4q^4(p^4+q^4) + 36p^6q^6\}(t^4+t^{-4})$$
$$+ \{6p^5q^5(p^2+q^2)\}(t^5+t^{-5})$$
$$+ p^6q^6(t^6+t^{-6}).$$

$$\phi(2, Y_1) = (p^4+q^4)$$
$$+ t^2\{4pq(p^2+q^2) + 4p^2q^2\}$$
$$+ t^4(2p^2q^2)$$

$$\phi(3, Y_1) = (p^6+q^6)$$
$$+ t^2\{4pq(p^4+q^4) + 2p^2q^2(p^2+q^2)\}$$
$$+ t^3\{2pq(p^4+q^4) + 4p^2q^2(p^2+q^2) + 6p^3q^3\}$$
$$+ t^4\{5p^2q^2(p^2+q^2) + 8p^3q^3\}$$
$$+ t^5\{4p^2q^2(p^2+q^2) + 4p^3q^3\}$$
$$+ t^7(2p^3q^3).$$

$$\phi(4, Y_1) = (p^8+q^8)$$
$$+ t^2\{4pq(p^6+q^6) + 2p^2q^2(p^4+q^4) + 2p^4q^4\}$$
$$+ t^3\{4pq(p^6+q^6)+4p^2q^2(p^4+q^4) + 4p^3q^3(p^2+q^2)\}$$
$$+ t^4\{8p^2q^2(p^4+q^4) + 12p^3q^3(p^2+q^2) + 8p^4q^4\}$$
$$+ t^5\{12p^2q^2(p^4+q^4) + 16p^3q^3(p^2+q^2) + 24p^4q^4\}$$
$$+ t^6\{2p^2q^2(p^4+q^4) + 12p^3q^3(p^2+q^2) + 20p^4q^4\}$$
$$+ t^7\{8p^3q^3(p^2+q^2) + 8p^4q^4\}$$
$$+ t^8\{4p^3q^3(p^2+q^2) + 6p^4q^4\}$$
$$+ t^{10}\{2p^4q^4\}.$$

$$\phi(5, Y_1) = (p^{10}+q^{10})$$
$$+ t^2\{4pq(p^8+q^8) + 2p^2q^2(p^6+q^6)$$
$$+ 2p^4q^4(p^2+q^2)\}$$

$$+ t^3 \{6pq (p^8+q^8) + 4p^2q^2 (p^6+q^6)$$
$$+ 4p^3q^3 (p^4+q^4) + 4p^5q^5\}$$
$$+ t^4 \{11p^2q^2 (p^6+q^6) + 12p^3q^3 (p^4+q^4)$$
$$+ 7p^4q^4 (p^2+q^2) + 8p^5q^5\}$$
$$+ t^5 \{20p^2q^2 (p^6+q^6) + 26p^3q^3 (p^4+q^4)$$
$$+ 28p^4q^4 (p^2+q^2) + 22p^5q^5\}$$
$$+ t^6 \{8p^2q^2 (p^6+q^6) + 32p^3q^3 (p^4+q^4)$$
$$+ 48p^4q^4 (p^2+q^2) + 48p^5q^5\}$$
$$+ t^7 \{24p^3q^3 (p^4+q^4) + 56p^4q^4 (p^2+q^2)$$
$$+ 64p^5q^5\}$$
$$+ t^8 \{20p^3q^3 (p^4+q^4) + 37p^4q^4 (p^2+q^2)$$
$$+ 56p^5q^5\}$$
$$+ t^9 \{2p^3q^3 (p^4+q^4) + 16p^4q^4 (p^2+q^2)$$
$$+ 32p^5q^5\}$$
$$+ t^{10} (12p^4q^4 (p^2+q^2) + 8p^5q^5\}$$
$$+ t^{11} \{4p^4q^4 (p^2+q^2) + 8p^5q^5\}$$
$$+ t^{13} \{2p^5q^5\}.$$

$$\phi(6, Y_1) = \{(p^{12}+q^{12})\}$$
$$+ t^2 \{4pq (p^{10}+q^{10}) + 2p^2q^2 (p^8+q^8)$$
$$+ 2p^4q^4 (p^4+q^4) + 2p^6q^6\}$$
$$+ t^3 \{8pq (p^{10}+q^{10}) + 4p^2q^2 (p^8+q^8)$$
$$+ 4p^3q^3 (p^6+q^6) + 4p^5q^5 (p^2+q^2)\}$$
$$+ t^4 \{14p^2q^2 (p^8+q^8) + 12p^3q^3 (p^6+q^6)$$
$$+ 8p^4q^4 (p^4+q^4) + 8p^5q^5 (p^2+q^2) + 8p^6q^6\}$$
$$+ t^5 \{28p^2q^2 (p^8+q^8) + 36p^3q^3 (p^6+q^6)$$
$$+ 28p^4q^4 (p^4+q^4) + 28p^5q^5 (p^2+q^2) + 24p^6q^6\}$$
$$+ t^6 \{18p^2q^2 (p^8+q^8) + 52p^3q^3 (p^6+q^6)$$

$$+ 72p^4q^4(p^4+q^4) + 56p^5q^5(p^2+q^2) + 66p^6q^6\}$$

$$+ t^7\{52p^3q^3(p^6+q^6) + 116p^4q^4(p^4+q^4)$$

$$+ 124p^5q^5(p^2+q^2) + 136p^6q^6\}$$

$$+ t^8\{52p^3q^3(p^6+q^6) + 109p^4q^4(p^4+q^4)$$

$$+ 196p^5q^5(p^2+q^2) + 186p^6q^6\}$$

$$+ t^9\{12p^3q^3(p^6+q^6) + 84p^4q^4(p^4+q^4)$$

$$+ 164p^5q^5(p^2+q^2) + 200p^6q^6\}$$

$$+ t^{10}\{46p^4q^4(p^4+q^4) + 108p^5q^5(p^2+q^2)$$

$$+ 154p^6q^6\}$$

$$+ t^{11}\{28p^4q^4(p^4+q^4) + 64p^5q^5(p^2+q^2)$$

$$+ 80p^6q^6\}$$

$$+ t^{12}\{2p^4q^4(p^4+q^4) + 20p^5q^5(p^2+q^2) + 48p^6q^6\}$$

$$+ t^{13}\{16p^5q^5(p^2+q^2) + 8p^6q^6\}$$

$$+ t^{14}\{4p^5q^5(p^2+q^2) + 10p^6q^6\}$$

$$+ t^{16}\{2p^6q^6\}.$$